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ANTI-SIMPSON'S QUADRATURE FORMULA AND ITS EXTENSION FOR EVALUATION OF ELLIPTIC AND OTHER INTEGRALS IN ADAPTIVE ENVIRONMENT

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Abstract: We have constructed an anti-Simpson's quadrature formula using Simpson's $\frac{1}{3}rd$ quadrature formula following the idea given by D. P. Laurie. An extension of this formula is developed by taking average linear combination with the Simpson's $\frac{1}{3}rd$ quadrature formula. Through error analysis, we studied the theoretical dominance of this extended anti-Simpson's quadrature formula over its constituents. We accomplished numerical verification of the formula evaluating test integrals including elliptic ones. We depict the novelty of the formula in both non-adaptive and adaptive environments. In adaptive environment the dominancy

of the rule over its constituents clarifies both in number of steps and error committed.

Keywords and Phrases: Simpson's $\frac{1}{3}rd$ quadrature formula $(Q_{S_3}(f))$, Anti-Simpson's 4-point quadrature formula $(Q_{aS_4}(f))$, Extended anti-Simpson's quadrature formula $(DS_1(f))$, Adaptive integration scheme, Elliptic integrals.

2020 Mathematics Subject Classification: Primary 65D30, 65D32, Secondary 65E05, 65A05.

1. Introduction

We have been using two methods, namely Richardson extrapolation and Kronord extension to produce higher precision formulae. We take trapezoidal quadrature formula and Gaussian quadrature formula as base formulae in Richardson extrapolation and Kronord extension respectively [1, 4, 3, 5, 8, 14]. These methods are not easy to handle. On the other hand, if we use mixed quadrature method to hike the precision, then we can easily accomplish this through very simple mathematical exercise [2, 9, 10, 11, 12, 13, 14, 15, 16].

In this paper, We wish to construct anti-Simpson's quadrature formula using the idea given by D. P. Laurie [7]. As Simpson's $\frac{1}{3}rd$ quadrature formula and anti-Simpson's 4-point quadrature formula are of same precision, we extend this anti-Simpson's quadrature formula by taking average linear combination with the Simpson's $\frac{1}{3}rd$ quadrature formula. The extended anti-Simpson's quadrature formula is a new way of precision hiking. The idea of anti-Gaussian quadrature was first used by Dirk P. Laurie [7]. Anti-Gaussian formula is (n+1) point quadrature formula. Its degree of precision is (2n-1) and it integrates all polynomials upto degree (2n+1) with an error equal in magnitude but of opposite in sign to that of n point Gaussian formula. If

$$Q_{aG_{n+1}}(f) = \sum_{i=1}^{n+1} \lambda_i f(x_i)$$

be (n+1)-point anti-Gaussian quadrature formula and $Q_{G_n}(f)$ the n point Gaussian formula, then

$$I(f) - Q_{aG_{n+1}}(f) = -\left(I(f) - Q_{G_n}(f)\right), f \in P_{2n+1} \tag{1}$$

where f is a polynomial of degree$\leq 2n+1$ and $I(f)$ is the exact value of the integral.

The contents of the paper are organized in the following manner. The section-1 is introductory one. In section-2, Based on the principle adopted by Laurie, we develop the anti-Simpson's quadrature formula from Simpson's $\frac{1}{3}rd$ quadrature formula and evaluate the error due to the formula. In section-3, Using the idea of mixed quadrature [2, 11, 12, 13, 14, 15, 16] an extension of the anti-Simpson's quadrature formula is developed by taking average linear combination with the Simpson's $\frac{1}{3}$ quadrature formula. Section-4, deals with error analysis showing the theoretical dominance of the extended anti-Simpson's quadrature formula over its constituent formulae. Section-5, includes numerical evaluation of some test integrals including elliptic ones using Simpson's $\frac{1}{3}rd$ quadrature formula, anti-Simpson's quadrature formula, Boole's quadrature formula, and extended anti-Simpson's quadrature formula in non-adaptive and adaptive mode. Section-7, reflects a brief conclusion about the potentiality of the extended anti-Simpson's formula.

2. Construction of anti-Simpson's quadrature formula

We choose Simpson's $\frac{1}{3}rd$ quadrature formula $Q_{S_3}(f)$:

$$I(f) = \int_{-1}^{1} f(x)dx \cong Q_{S_3}(f) = \frac{1}{3}\left[f(-1) + 4f(0) + f(1)\right] \tag{2}$$

We assume that anti-Simpson's quadrature formula due to Simpson's $\frac{1}{3}rd$ quadrature formula is a 4-point formula whose degree of precision is same as that of the Simpson's $\frac{1}{3}rd$ quadrature formula i.e. 3. It is of the following form

$$Q_{aS_4}(f) = \sum_{i=1}^{4} \lambda_i f(x_i) \tag{3}$$

Simpson's $\frac{1}{3}rd$ quadrature formula is a closed type quadrature formula. Hence, anti-Simpson's 4-point quadrature formula eqn.(3) due to Simpson's $\frac{1}{3}rd$ quadrature formula becomes

$$Q_{aS_4}(f) = \lambda_1 f(-1) + \lambda_2 f(x_2) + \lambda_3 f(x_3) + \lambda_4 f(1) \tag{4}$$

Here $x_1 = -1, x_4 = 1$ and having six unknowns namely λ_i for $i = 1, 2, 3, 4$ and x_i for $i = 2, 3$.

To determine these unknowns we use the Laurie's condition eqn.(**??**) for n=3 to get

$$I(f) - Q_{aS_4}(f) = -\left(I(f) - Q_{S_3}(f)\right)$$

for $f(x) = x^i, i = 0, 1, 2, 3, 4, 5$.

$$\Rightarrow Q_{aS_4}(f) = 2I(f) - Q_{S_3}(f) \tag{5}$$

In order to evaluate λ_i, for $i = 1, 2, 3, 4$ and x_i for i=2,3 with $f(x) = x^i, i = 0, 1, 2, 3, 4, 5$, we have following six equations

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2 \tag{6}$$
$$-\lambda_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 = 0 \tag{7}$$
$$\lambda_1 + \lambda_2 (x_2)^2 + \lambda_3 (x_3)^2 + \lambda_4 = 2/3 \tag{8}$$
$$-\lambda_1 + \lambda_2 (x_2)^3 + \lambda_3 (x_3)^3 + \lambda_4 = 0 \tag{9}$$
$$\lambda_1 + \lambda_2 (x_2)^4 + \lambda_3 (x_3)^4 + \lambda_4 = 2/15 \tag{10}$$
$$-\lambda_1 + \lambda_2 (x_2)^5 + \lambda_3 (x_3)^5 + \lambda_4 = 0 \tag{11}$$

Solving the system of equations we obtain, $\lambda_1 = -\frac{1}{9} = \lambda_4, \lambda_2 = \frac{10}{9} = \lambda_3, x_2 = \sqrt{\frac{2}{5}}$ and $x_3 = -\sqrt{\frac{2}{5}}$. Putting the above values in eqn.(4), we have

$$Q_{aS_4}(f) = \frac{1}{9}\left[10\left\{f\left(\sqrt{\frac{2}{5}}\right) + f\left(-\sqrt{\frac{2}{5}}\right)\right\} - (f(-1) + f(1))\right] \tag{12}$$

Eqn. (12) is the required anti-Simpson's 4-point quadrature formula.

Theorem 1. *Let f(x) contains derivative of all orders in the closed interval* $[-1, 1]$. *Then the error* $E_{aS_4}(f)$ *associated with the formula* $Q_{aS_4}(f)$ *is given by* $|E_{aS_4}(f)| \cong \frac{1}{90} \mid f^{iv}(0) \mid$

Proof. Since $f(x)$ is sufficiently differentiable function in the closed interval $[-1, 1]$, using Maclaurin's expansion of $f(x)$ one can obtain

$$I(f) = \int_{-1}^{1} f(x)dx = 2f(0) + \frac{1}{3}f^{ii}(0) + \frac{2}{5!}f^{iv}(0) + \frac{2}{7!}f^{vi}(0) + \cdots \tag{13}$$

and

$$Q_{aS_4}(f) = 2f(0) + \frac{1}{3}f^{ii}(0) + \frac{2}{15 \times 4!}f^{iv}(0) - \frac{2}{25 \times 6!}f^{vi}(0) + \cdots \tag{14}$$

Using eqn. (13) and eqn. (14), Error associated with anti-Simpson's 4-point quadrature formula is

$$E_{aS_4}(f) = I(f) - Q_{aS_4}(f)$$

or

$$E_{aS_4}(f) = \frac{1}{90}f^{iv}(0) + \frac{64}{7! \times 25}f^{vi}(0) + \cdots \tag{15}$$

So $|E_{aS_4}(f)| \cong \frac{1}{90} \mid f^{iv}(0) \mid$.

3. Extension of anti-Simpson's quadrature formula

We choose the anti-Simpson's 4-point quadrature formula $Q_{aS_4}(f)$

$$I(f) \cong Q_{aS_4}(f) = \frac{1}{9}\left[10\left\{f(\sqrt{\frac{2}{5}}) + f(-\sqrt{\frac{2}{5}})\right\} - \{f(-1) + f(1)\}\right] \tag{16}$$

and the Simpson's $\frac{1}{3}$ quadrature formula $Q_{S_3}(f)$

$$I(f) \cong Q_{S_3}(f) = \frac{1}{3}\left[f(-1) + 4f(0) + f(1)\right] \tag{17}$$

From eqn. (16) and eqn. (17), we get

$$I(f) = Q_{aS_4}(f) + E_{aS_4}(f) \tag{18}$$

and

$$I(f) = Q_{S_3}(f) + E_{S_3}(f) \tag{19}$$

where

$$E_{aS_4}(f) = \frac{1}{90}f^{iv}(0) + \frac{64}{25 \times 7!}f^{vi}(0) + \cdots \tag{20}$$

$$E_{S_3}(f) = \frac{-1}{90}f^{iv}(0) - \frac{8}{3 \times 7!}f^{vi}(0) - \cdots \tag{21}$$

$E_{aS_4}(f)$ and $E_{S_3}(f)$ are truncation error due to Q_{aS_4} and $Q_{S_3}(f)$ respectively. The extended anti-Simpson's 4-point quadrature formula denoted by $DS_1(f)$ is obtained by taking average linear combination of constituent formulas $Q_{aS_4}(f)$ and $Q_{S_3}(f)$ as follows.

Adding eqn. (18) and eqn. (19), we obtain

$$2I(f) = [Q_{aS_4}(f) + Q_{S_3}(f)] + [E_{aS_4}(f) + E_{S_3}(f)]$$

or

$$I(f) = \frac{1}{2}\left[Q_{aS_4}(f) + Q_{S_3}(f)\right] + \frac{1}{2}\left[E_{aS_4}(f) + E_{S_3}(f)\right]$$

$$\Rightarrow I(f) = DS_1(f) + EDS_1(f)$$

where

$$DS_1(f) = \frac{1}{2}\left[Q_{aS_4}(f) + Q_{S_3}(f)\right] \tag{22}$$

or

$$DS_1(f) = \frac{1}{9}\left[f(-1) + 5f\left(-\sqrt{\frac{2}{5}}\right) + 6f(0) + 5f\left(\sqrt{\frac{2}{5}}\right) + f(1)\right] \tag{23}$$

and

$$EDS_1(f) = \frac{1}{2}\left[E_{aS_4}(f) + E_{S_3}(f)\right] \tag{24}$$

Eqn. (22), expresses the desired extended anti-Simpson's quadrature formula and eqn. (24) represents the error generated due to this quadrature formula. The formula eqn. (23) is called extended anti-Simpson's quadrature formula as it is developed from the anti-Simpson's quadrature formula.
Substituting eqn. (20) and eqn. (21) into eqn. (24), *Error associated with the extended anti-Simpson's quadrature formula*

$$EDS_1(f) = \frac{-4}{75 \times 7!} f^{vi}(0) - \cdots \tag{25}$$

We see that first term of $EDS_1(f)$ contains 6th order derivative of the integrand, thus the formula is exact for all polynomials of degree≤ 5. So $DS_1(f)$ is of precision 5.

4. Error analysis of the Extended anti-Simpson's quadrature formula

Asymptotic error estimate of the extended anti-Simpson quadrature formula (eqn. (23)) is given in theorem 1.

Theorem 2. *Let $f(x)$ contains derivatives of all orders in the closed interval $[-1, 1]$. Then the error $EDS_1(f)$ associated with the rule $DS_1(f)$ is given by* $|EDS_1(f)| \cong \frac{4}{75\times7!} \mid f^{vi}(0) \mid$

Proof. We know that from eqn. (25) $EDS_1(f) = \frac{-4}{75\times7!} f^{vi}(0) - \cdots$
so $\mid EDS_1(f) \mid \cong \frac{4}{75\times7!} \mid f^{vi}(0) \mid$

The error bound of the formula eqn. (24) is given in the theorem 3.

Theorem 3. *The error bound of the truncation error for $DS_1(f)$ is given by*

$$EDS_1(f) \mid \leq \frac{M}{180} \mid \eta_2 - \eta_1 \mid, \ \eta_1, \eta_2 \in [-1, 1]$$

where $M = \max_{-1\leq x\leq 1} \mid f^{v}(x) \mid$

Proof. We have

$$E_{aS_4}(f) \cong \frac{1}{90} f^{iv}(\eta_2), \eta_2 \in [-1, 1]$$

$$E_{S_3}(f) \cong -\frac{1}{90} f^{iv}(\eta_1), \eta_1 \in [-1, 1]$$

Now

$$EDS_1(f) = \frac{1}{2}\left[E_{aS_4}(f) + E_{S_3}(f)\right]$$

$$= \frac{1}{180}\left[f^{iv}(\eta_2) - f^{iv}(\eta_1)\right]$$

$$= \frac{1}{180}\int_{\eta_1}^{\eta_2} f^{v}(x)dx, (assuming\, \eta_1 < \eta_2)$$

From this we obtain

$$\mid EDS_1(f) \mid = \frac{1}{180} \mid \int_{\eta_1}^{\eta_2} f^{v}(x)dx \mid \leq \frac{1}{180}\int_{\eta_1}^{\eta_2} \mid f^{v}(x) \mid dx$$

So $\mid EDS_1(f) \mid \leq \frac{M}{180} \mid \eta_2 - \eta_1 \mid$.

This shows that the error bound will become smaller if η_1, η_2 are take sufficiently closure to each other.

Corollary. *The error bound of* $DS_1(f)$ *is given by* $\mid EDS_1(f) \mid \leq \frac{M}{90}$, *where* $M = \max\limits_{-1\leq x\leq 1} \mid f^{v}(x) \mid$

Proof. We know from the theorem-3, that

$$EDS_1(f) \mid \leq \frac{M}{180} \mid \eta_2 - \eta_1 \mid,\ \eta_1, \eta_2 \in [-1, 1]$$

where $M = \max\limits_{-1\leq x\leq 1} \mid f^{v}(x) \mid$

choosing, $\mid \eta_1 - \eta_2 \mid \leq 2$, we have $\mid EDS_1(f) \mid \leq \frac{M}{90}$.

4.1. Adaptive Integration Scheme using $DS_1(f)$ quadrature formula

An adaptive integration scheme adopted in this paper is designed [3, 4, 6, 9, 10, 11, 12, 13, 14] using $DS_1(f)$ and its constituents formulae.

5. Numerical Verifications

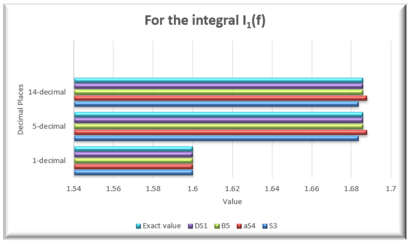
5.1. Observation

The results shown in the table-1 & table-2 are diagrammatically represented in figure-1 & figure-2 respectively. It is observed in table-1 that the precision of Simpson $\frac{1}{3}$rd rule $Q_{S_3}(f)$ and anti-Simpson 4-point rule $Q_{aS_4}(f)$ are same but due to the construction principle of $Q_{aS_4}(f)$, the results of the most of the test integrals are nearly equal when computations are made in non-adaptive mode. But as observed in table-2, the accuracy of results due to extended anti-Simpson rule $DS_1(f)$ is slightly edge over the existing Boole's 5-point rule $Q_{B_5}(f)$ in non-adaptive environment.

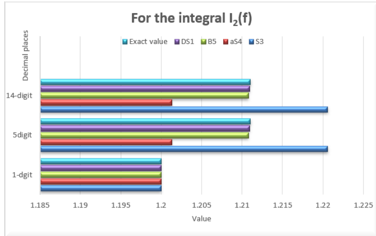
On other hand, the story is some what encouraging in adaptive environment. Though the rules $DS_1(f)$ and $Q_{B_5}(f)$ are having identical precision, as observed in table-4 and figure-1(d), in computational results of most of the test integrals

Table 1: Approximation of some elliptic integrals and other integrals using Simpson's $\frac{1}{3}rd$ 3quadrature formula $Q_{S_3}(f)$, and anti-Simpson's 4-point quadrature formula $Q_{aS_4}(f)$ in non adaptive integration scheme

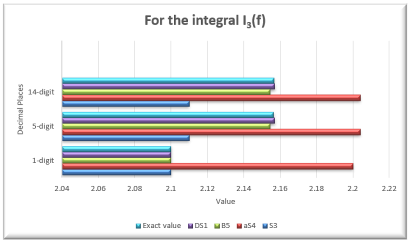
			Approximate value $Q(f)$	
Sl.No	Integral	Exact value	$Q_{S_3(f)}$	$Q_{aS_4}(f)$
I_1	$\int_0^{\frac{\pi}{2}}(1-0.25sin^2x)^{\frac{-1}{2}}dx$	1.685750354812596	1.683600554074	1.687934008653
I_2	$\int_0^{\frac{\pi}{2}}(1-(1-0.25)sin^2x)^{\frac{1}{2}}dx$	1.211056025	1.22058143717914	1.201319079852459
I_3	$\int_0^{\frac{\pi}{2}}(1-(1-0.25)sin^2x)^{-\frac{1}{2}}dx$	2.156515647499643	2.1100099321617	2.20404202102847
I_4	$\int_0^{\frac{\pi}{2}}(1-0.65sin^2x)^{\frac{1}{2}}dx$	1.270707480	1.27704257576664	1.26424022707844
I_5	$\int_0^{\frac{\pi}{2}}(1-0.65sin^2x)^{\frac{-1}{2}}dx$	2.007598398424376	1.978929389546905	2.037036015678938
I_6	$\int_0^{\frac{\pi}{2}}(1-(1-0.65)sin^2x)^{\frac{1}{2}}dx$	1.422691133	1.42403321308067	1.42132806103853
I_7	$\int_0^{\frac{\pi}{2}}(1-(1-0.65)sin^2x)^{-\frac{1}{2}}dx$	1.744350597225613	1.739448122402486	1.7493397885902
I_8	$\int_0^1 \sqrt{x}dx$	$0.66666666666\cdots$	0.6380711874576	0.68452215750627
I_9	$\int_0^{\frac{\pi}{2}} \frac{1}{sin^2x+\frac{1}{4}cos^2x}dx$	$\pi \cong 3.14159265358979$	2.98451302090592	3.30078360295501
I_{10}	$\int_0^1 x^{16}cos(x^{16})dx$	0.0491217295	0.090060556	-0.0084700278



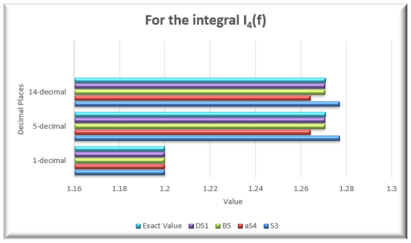
(a) Graphical analysis of values for $I_1(f)$ obtain by different quadrature rules as reflected in the table 1 and table 2



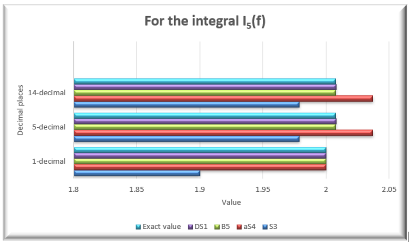
(b) Graphical analysis of values for $I_2(f)$ obtain by different quadrature rules as reflected in the table 1 and table 2



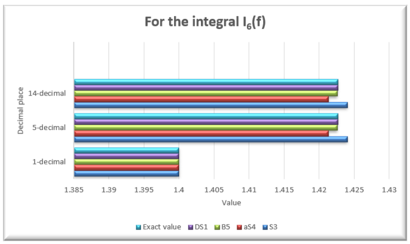
(c) Graphical analysis of values for $I_3(f)$ obtain by different quadrature rules as reflected in the table 1 and table 2



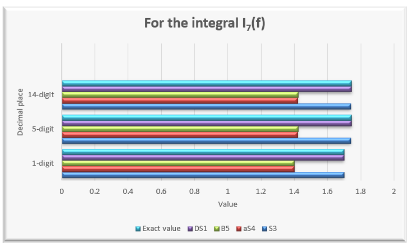
(d) Graphical analysis of values for $I_4(f)$ obtain by different quadrature rules as reflected in the table 1 and table 2



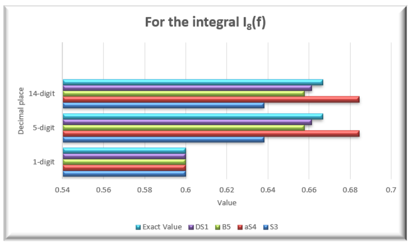
(e) Graphical analysis of values for $I_5(f)$ obtain by different quadrature rules as reflected in the table 1 and table 2



(f) Graphical analysis of values for $I_6(f)$ obtain by different quadrature rules as reflected in the table 1 and table 2



(g) Graphical analysis of values for $I_7(f)$ obtain by different quadrature rules as reflected in the table 1 and table 2

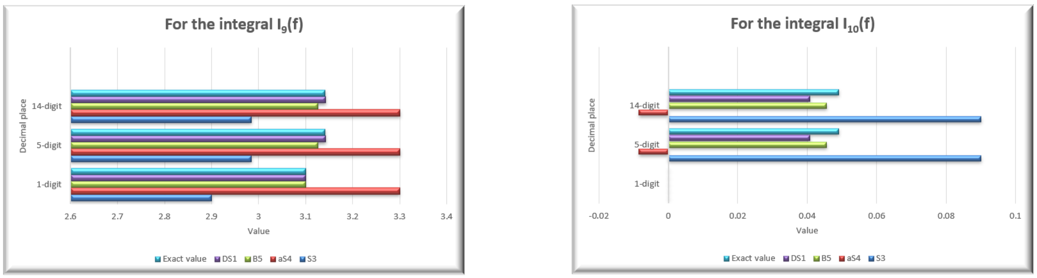


(h) Graphical analysis of values for $I_8(f)$ obtain by different quadrature rules as reflected in the table 1 and table 2

Figure 1: Graphical analysis of value of $DS_1(f)$ and its constituents for the integrals $I_1(f)$ to $I_8(f)$

Table 2: Approximation of integrals (as given in the table-1) Boole's quadrature formula $Q_{B_5}(f)$ and anti-Simpson's 5-point quadrature formula $DS_1(f)$ in non-adaptive integration scheme

		Approximate value $Q(f)$	
Sl. No	Integral	$Q_{B_5}(f)$	$DS_1(f)$
1	I_1	1.68588495654325	1.68576728136386
2	I_2	1.2108041907406	1.2109502585158
3	I_3	2.15445682517894	2.1570259765951
4	I_4	1.270431508053661	1.27064140142254
5	I_5	2.0078103540157	2.00798270261292
6	I_6	1.422606859567265	1.422680637059604
7	I_7	1.74463240875653	1.74439395549634
8	I_8	0.65775660328156	0.6612966724819
9	I_9	3.125501569261942	3.142648311930468
10	I_{10}	0.0455889578	0.04079526448



(a) Graphical analysis of values for $I_9(f)$ obtain by different quadrature rules as reflected in the table 1 and table 2

(b) Graphical analysis of values for $I_{10}(f)$ obtain by different quadrature rules as reflected in the table 1 and table 2

Figure 2: Graphical analysis of value of $DS_1(f)$ and its constituents for the integrals $I_9(f)$ and $I_{10}(f)$

Table 3: Approximation of integrals (as given in the table-1) using Simpson's $\frac{1}{3}rd$ quadrature formula $Q_{S_3}(f)$, anti-Simpson's 4-point quadrature formula $Q_{aS_4}(f)$ in adaptive integration scheme

	Approximate value $Q(f)$					
Integral	$Q_{S_3}(f)$	no. of st.	Error	$Q_{aS_4}(f)$	no. of st.	Frror
I_1	1.68575035480694192	15	5.654×10^{-12}	1.68575035480694191	15	5.654×10^{-12}
I_2	1.21105611607538	17	9.107×10^{-8}	1.211055939043779	17	8.595×10^{-8}
I_3	2.1565156801508667	23	3.265×10^{-8}	2.156515614820341	23	3.267×10^{-8}
I_4	1.270707532083227	17	5.208×10^{-8}	1.270707427205607	17	5.279×10^{-8}
I_5	2.007598389539456	23	8.884×10^{-9}	2.007598407286577	23	8.862×10^{-9}
I_6	1.422691130264387	13	2.735×10^{-9}	1.4226911366763836	13	3.676×10^{-9}
I_7	1.74435056618554	13	3.104×10^{-8}	1.744350628256197	17	3.103×10^{-8}
I_8	0.666666424721	35	2.419×10^{-7}	0.66666699263446	33	3.259×10^{-7}
I_9	3.1415926364045382	41	1.717×10^{-8}	3.141592670762019	41	1.718×10^{-8}
I_{10}	0.04912175159406	29	2.209×10^{-8}	0.0491217075943	29	2.1905×10^{-8}

Table 4: Approximation of integrals (as given in the table-1) using Boole's quadrature formula $Q_{B_5}(f)$ and anti-Simpson's 5-point quadrature formula $DS_1(f)$ in adaptive integration scheme

	Approximate value $Q(f)$					
Integral	$Q_{B_5}(f)$	no. of st.	Error	$DS_1(f)$	no. of st.	Frror
I_1	1.6857503554515023	05	6.389×10^{-10}	1.68575035480803807	03	4.557×10^{-12}
I_2	1.2110560275608	07	2.5608×10^{-9}	1.21105602949774084	05	4.4977×10^{-9}
I_3	2.15651564179088	09	5.708×10^{-9}	2.15651564752361873	07	2.3975×10^{-11}
I_4	1.2707074850734	05	5.073×10^{-9}	1.27070748048496923	05	4.849×10^{-10}
I_5	2.00759839842057978	07	3.796×10^{-12}	2.00759839841657174	07	7.804×10^{-12}
I_6	1.42269113347251735	03	4.725×10^{-10}	1.42269113348541222	03	4.854×10^{-10}
I_7	1.74435059721953973	07	6.073×10^{-12}	1.74435059724776781	03	2.215×10^{-11}
I_8	0.666666383126315876	19	2.835×10^{-7}	0.6666665008478	19	1.658×10^{-7}
I_9	3.14159247674623714	09	1.768×10^{-7}	3.14159265292398245	11	6.5601×10^{-10}
I_{10}	0.049121737163137	13	7.663×10^{-9}	0.04912173720359629	09	7.7035×10^{-9}

the new rule $DS_1(f)$ is better so far as the number of steps and the computational errors are concerned. Further, if we compare the results of figure-1(c), the new rule $DS_1(f)$ is much more dominating than the constituent rules $Q_{S_3}(f)$ and $Q_{aS_4}(f)$.

6. Conclusion

Analysing the observation, we conclude that the extended anti-Simpson's rule $DS_1(f)$ dominates both its constituent rules $Q_{S_3}(f)$ and $Q_{aS_4}(f)$ in non-adaptive mode.The visibility of this dominance is much more in adaptive environment. Though the precision of existing Boole's 5-point $Q_{B_5}(f)$ and that $DS_1(f)$ is same, clear dominance is seen in case of $DS_1(f)$ in adaptive mode. Therefore, $DS_1(f)$ may be adopted as an alternative rule of integration.

Advantage. The precision extension in this method is very simple and straightforward unlike Kronord extension.

Scope. The extended quadrature based on anti-Simpson formula can further be extended using anti-Lobatto and anti-Gaussian formulae.

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